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The Compromise Value for NTU-Games

P. Borm[1], H. Keiding[2], R. P. McLean[3], S. Oortwijn[4], and S. Tijs[1]

Abstract: The compromise value is introduced as a single-valued solution concept for NTU-games. It is shown that the compromise value coincides with the τ-value for TU-games and with the Kalai-Smorodinsky solution for bargaining problems. In addition the axiomatic characterizations of both the two-person Kalai-Smorodinsky solution and the τ-value can be extended to the compromise value for large classes of NTU-games.

We also present an alternative NTU-extension of the TU τ-value (called the NTU τ-value) which coincides with the Nash solution for two-person bargaining problems. The definition of the NTU τ-value is analogous to that of the Shapley NTU-value.

Both the compromise value and the NTU τ-value are illustrated by means of the Roth-Shafer examples.

1 Introduction

The Shapley value of TU (= Transferable Utility)-games, introduced by Shapley (1953), has been generalized to NTU (= Non Transferable Utility)-games in various ways. Shapley (1969) defined the NTU-value and Harsanyi (1959, 1963), Owen (1971) and Imai (1983) considered other possible extensions. For the NTU-value an axiomatic characterization has been provided by Aumann (1985a) and Kern (1985), for the Harsanyi solution by Hart (1985a) and for monotonic solutions by Kalai and Samet (1985).

This paper introduces the compromise value as an extension of the τ-value of Tijs (1981) for quasi-balanced TU-games to the class of compromise admissible NTU-games.

The compromise value as defined in section 3 is a one-point solution concept that is based upon the upper and lower bounds for the core of an NTU-game that are given in section 2. Interestingly, the compromise value coincides with the solution of Kalai and Smorodinsky (1975, in short KS-solution) for the special case of bargaining games.

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Section 4 shows that both the axiomatic characterization of the 2-person KS-solution and the axiomatic characterization of the τ-value given by Tijs (1987) can be extended to the compromise value on a class of NTU-games. As a result, the compromise value is an extension of the KS-solution to NTU-games from a definitional as well as an axiomatic viewpoint. This may at first seem rather confusing if one recalls the paper of Roth (1979) showing, by means of an example, that there can be no solution for general n-person bargaining problems which satisfies the (analogues of the) axioms of the 2-person KS-solution. Roth's example will be discussed in detail and, as we will see, the issue here is comprehensiveness in combination with a weaker version of Pareto optimality.

Roth (1980) and Shafer (1980) introduced two special classes of games for which, in their opinion, the Shapley NTU-value leads to a counterintuitive outcome. This led to an interesting discussion of the NTU-value in the papers of Aumann (1985b, 1986), Roth (1986) and Hart (1985b). In Section 5 the Roth-Shafer examples are discussed in some detail and we compare the compromise value to the Shapley and Harsanyi NTU-values for these examples.

Section 6 briefly discusses some results on the NTU τ-value which is defined along the lines of the Shapley NTU-value. Among others, it is seen that the NTU τ-value coincides with the Nash solution for 2-person bargaining problems and that one can prove existence (i.e. non-emptiness) for (a subclass of) compactly and convexly generated NTU-games.

Notation: Let $x, y \in \mathbb{R}^n$ and $C, D \subset \mathbb{R}^n$. We have $x \geq (>) y$ if and only if $x_i \geq (>) y_i$ for all $i \in \{1, \ldots, n\}$, $\mathbb{R}^n_+ := \{z \in \mathbb{R}^n \mid z \geq 0\}$ and $\mathbb{R}^n_{++} := \{z \in \mathbb{R}^n \mid z > 0\}$, $xy := \sum_{i=1}^{n} x_i y_i \in \mathbb{R}$, $x * y := (x_1 y_1, \ldots, x_n y_n) \in \mathbb{R}^n$, $x * C := \{x * c \in \mathbb{R}^n \mid c \in C\}$ and $C + D := \{c + d \in \mathbb{R}^n \mid c \in C, d \in D\}$.
Further, $Conv(C)$ denotes the convex hull of C and

$$Comp(C) := \{z \in \mathbb{R}^n \mid \text{ there is a } c \in C \text{ such that } z \leq c\}$$

is the comprehensive hull of C.
Finally, with $N := \{1, \ldots, n\}$ and $i \in N$, $e_i \in \mathbb{R}^n$ denotes the i-th unit vector and $e_N = \sum_{i \in N} e_i$ is the n-tuple of 1's, and for $S \subset N$, $x_S := (x_j)_{j \in S} \in \mathbb{R}^S$ and x is identified with $(x_S, x_{N \setminus S})$.

2 NTU-Games: Bounds for the Core

An *NTU-game* is a pair (N, V) where $N := \{1, 2, \ldots, n\}$ is the set of players and V is a set-valued function that assigns to each *coalition* $S \in 2^N \setminus \{\emptyset\}$ a non-empty set $V(S) \subset \mathbb{R}^S$ of *attainable* payoff vectors. For each player $i \in N$ we assume there is a *individual rational* payoff $v(i) \in \mathbb{R}$ such that $V(\{i\}) = \{a \in \mathbb{R} \mid a \leq v(i)\}$ while, for each $S \in 2^N \setminus \{\emptyset\}$,

(i) $V(S)$ is closed and comprehensive (i.e. if $a \in V(S)$ and $b \in \mathbb{R}^S$ is such that $b \leq a$, then $b \in V(S)$).
(ii) $V(S) \cap \{a \in \mathbb{R}^S \mid a \geq (v(j))_{j \in S}\}$ is bounded.

An NTU-game (N, V) will be often identified with V. The *core* $C(V)$ consists of those attainable payoff vectors for the grand coalition N which are stable with respect to (strict) domination. More specifically, with

$$Dom(V(S)) := \{a \in \mathbb{R}^S \mid \exists_{b \in V(S)}: b > a\} \tag{1}$$

representing the set of dominated payoff vectors for a coalition $S \in 2^N \setminus \{\emptyset\}$,

$$C(V) := \{a \in V(N) \mid \neg \exists_{S \in 2^N \setminus \{\emptyset\}}: a_S \in Dom(V(S))\}. \tag{2}$$

Let $i \in N$. Assuming that the coalition $N \setminus \{i\}$ will never agree to a payoff vector $a \in \mathbb{R}^{N \setminus \{i\}}$ with $a \in Dom(V(N \setminus \{i\}))$ or $a_j < v(j)$ for some $j \in N \setminus \{i\}$, the highest possible marginal contribution of player i by joining the coalition $N \setminus \{i\}$ is given by

$$K_i(V) := \sup \{t \in \mathbb{R} \mid \exists_{a \in \mathbb{R}^{N \setminus \{i\}}}: (a, t) \in V(N),\ a \notin Dom(V(N \setminus \{i\})) \text{ and } a \geq (v(j))_{j \in N \setminus \{i\}}\}. \tag{3}$$

$K_i(V)$ is called the *utopia payoff* to player i. By assumption (ii) in the definition of an NTU-game we have that $K_i(V) < \infty$. However, $K_i(V) = -\infty$ might occur.

Assume $K_j(V) \in \mathbb{R}$ for all $j \in N$ and consider a coalition S to which player i belongs. The formation of such a coalition is attractive for a player $j \in S \setminus \{i\}$ if he gets (slightly) more that the utopia payoff $K_j(V)$. Thus, player i can lay a rightful claim to the *remainder* $\rho_i^S(V)$ which is given by

$$\rho_i^S(V) := \sup \{t \in \mathbb{R} \mid \exists_{a \in \mathbb{R}^{S \setminus \{i\}}}: (t, a) \in V(S) \text{ and } a > K_{S \setminus \{i\}}(V)\}. \tag{4}$$

Among the 2^{n-1} possible coalitions with $i \in S$, player i can choose one where this remainder is maximal. Let

$$k_i(V) := \max_{S: i \in S} \rho_i^S(V) \tag{5}$$

denote the *minimal right* of player i. Clearly $k_i(V) \geq v(i)$, but it might occur that $k_i(V) = \infty$. In this paper we concentrate on NTU-games for which all utopia payoffs and minimal rights for the various players are real numbers. In particular, this is the case for NTU-games with a non-empty core: Theorem 1 shows that $K(V) = (K_j(V))_{j \in N}$ and $k(V) = (k_j(V))_{j \in N}$ establish an upper and lower bound for the core, respectively.

Theorem 1: Let (N, V) be an NTU-game with $x \in C(V)$. Then

$$k(V) \leq x \leq K(V).$$

Proof: Obviously, (2) and (3) imply

$$K_j(V) \geq \sup\{t \in \mathbb{R} \mid \exists_{a \in \mathbb{R}^{N \setminus \{j\}}}: (a, t) \in C(V)\} \geq x_j$$

for all $j \in N$. Hence, $x \leq K(V)$.

Let $i \in N$ and choose a coalition $T \ni i$ such that $k_i(V) = \rho_i^T(V) = \max_{S: i \in S} \rho_i^S(V)$. Suppose $k_i(V) > x_i$. Then we can choose $\varepsilon > 0$ such that $k_i(V) > x_i + \varepsilon$. Further, by (4), there exists a vector $a \in \mathbb{R}^{T \setminus \{i\}}$ such that $(x_i + \varepsilon, a) \in V(T)$ and $a > K_{T \setminus \{i\}}(V)$. However, this would imply that

$$x_T \leq (x_i, K_{T \setminus \{i\}}(V)) < (x_i + \varepsilon, a) \in V(T),$$

which contradicts the fact that $x \in C(V)$. Hence, $k(V) \leq x$. □

The vectors $k(V)$ and $K(V)$ induce familiar bounds for TU-games and two-person bargaining games.

(a) *TU-games.* A *TU-game* is a pair (N, v) where v is a function that assigns to each coalition S a real number $v(S)$ with $v(\emptyset) = 0$. The core $C(v)$ is defined by

$$C(v) := \{a \in \mathbb{R}^N \mid \sum_{i \in N} a_i = v(N),\ \sum_{i \in S} a_i \geq v(S) \text{ for all } S \subset N\}.$$

For a TU-game (N, v), Tijs (1981) introduced a utopia vector $M(v) \in \mathbb{R}^N$ and a minimal right vector $m(v) \in \mathbb{R}^N$ as follows. For $i \in N$,

$$M_i(v) := v(N) - v(N \setminus \{i\}) \text{ and } m_i(V) := \max_{S: i \in S} (v(S) - \sum_{j \in S \setminus \{i\}} M_j(v)). \tag{6}$$

For $x \in C(v)$, it was shown that $m(v) \leq x \leq M(v)$.
Associating an NTU-game (N, V) to a TU-game (N, v) by defining

$$V(S) := \{a \in \mathbb{R}^S \mid \sum_{i \in S} a_i \leq v(S)\} \tag{7}$$

for all $S \in 2^N \setminus \{\emptyset\}$, it is straightforward to verify that $C(v) = C(V)$, and that $M(v) = K(V)$ and $m(v) = k(V)$ if v is such that $v(N \setminus \{i\}) \geq \sum_{j \in N \setminus \{i\}} v(j)$ for all $i \in N$.

(b) *Bargaining problems.* In an n-person *bargaining problem* (C, d), the non-empty set $C \subset \mathbb{R}^n$ represents the set of feasible outcomes and $d \in C$ is the disagreement point. Moreover, we assume that the following properties are satisfied:

(i) C is closed, convex and comprehensive.
(ii) There is an $x^0 \in C$ with $x^0 > d$.
(iii) $C_d := \{x \in C \mid x \geq d\}$ is bounded.

For a bargaining problem (C, d), Kalai and Smorodinsky (1975) introduced the utopia point $u(C, d) \in \mathbb{R}^N$ by defining $u_i(C, d) := \max\{a \in \mathbb{R} \mid \exists_{b \in \mathbb{R}^{N \setminus \{i\}}} : (a, b) \in C_d\}$ for all $i \in N$. Each bargaining problem (C, d) corresponds to an NTU-game (N, V) defined by setting $V(N) = C$ and $V(S) = \{a \in \mathbb{R}^S \mid a \leq d_S\}$ for $S \subset N$, $S \neq \emptyset$. Then, one easily obtains that $u(C, d) = K(V)$ and $d = k(V)$.

3 The Compromise Value

In this section the compromise value is introduced as an extension of the τ-value for quasi-balanced TU-games (cf. Tijs (1981)) to the class of compromise admissible NTU-games.

Here, an NTU-game (N, V) is called *compromise admissible* if the utopia vector $K(V)$ and the minimal right vector $k(V)$ of section 2 satisfy the following two properties:

(i) $k(V) \leq K(V)$.
(ii) $k(V) \in V(N)$, $K(V) \notin Dom(V(N))$.

By $\mathscr{C}^N$ we denote the class of all compromise admissible NTU-games with player set N. Clearly, we have

Lemma 2: Every NTU-game with a non-empty core is compromise admissible.

Proof: Let (N, V) be an NTU-game with $x \in C(V)$. Then, using theorem 1, $k(V) \leq x \leq K(V)$. In particular, since $x \in V(N)$, comprehensiveness implies that $k(V) \in V(N)$.

Suppose $K(V) \in Dom(V(N))$. Then there is an $y \in V(N)$ such that $y > K(V) \geq x$. However, this contradicts the fact that $x \in C(V)$. We may conclude that the conditions (i) and (ii) are satisfied. □

For $V \in \mathscr{C}^N$ the *compromise value* $T(V) \in \mathbb{R}^N$ is defined as the unique vector on the line segment between $k(V)$ and $K(V)$ which lies in $V(N)$ and is closest to the utopia vector $K(V)$. More specifically,

$$T(V) := \lambda_V K(V) + (1 - \lambda_V) k(V), \tag{8}$$

where

$$\lambda_V := \max\{\lambda \in [0, 1] \mid \lambda K(V) + (1 - \lambda) k(V) \in V(N)\}. \tag{9}$$

Note that λ_V is well-defined because $k(V) \in V(N)$ and $V(N)$ is closed and comprehensive.

Using the notations of section 2, a TU-game (N, v) is called *quasi-balanced* if $m(v) \leq M(v)$ and $\sum_{i \in N} m_i(v) \leq v(N) \leq \sum_{i \in N} M_i(v)$. For a quasi-balanced TU-game (N, v) the *τ-value* $\tau(v) \in \mathbb{R}^N$ is defined as the unique vector lying on the line segment between $m(v)$ and $M(v)$ which is efficient, i.e. $\sum_{i \in N} \tau_i(v) = v(N)$.

Assume $v(N \setminus \{i\}) \geq \sum_{j \in N \setminus \{i\}} v(j)$ for all $i \in N$ and let V be the NTU-game corresponding to v (cf. (7)). One easily verifies that v is quasi-balanced if and only if V is compromise admissible, and that the τ-value of v coincides with the compromise value of V.

For a bargaining problem (C, d) one finds that the compromise value of the corresponding NTU-game V is the unique undominated feasible outcome lying on the line segment between the disagreement point d and the utopia point $u(C, d)$. In particular, this outcome corresponds to the KS-solution for the bargaining problem (C, d).

4 Axiomatic Characterizations

As we have seen, the compromise value 'definitionally' extends the Kalai-Smorodinsky solution to NTU-games. In the first part of this section we show that the axiomatic characterization of the 2-person KS-solution can be extended to the n-person NTU-case as well.

Recalling the axioms of Kalai and Smorodinsky (1975), we first have to note that they used a slightly different definition of a bargaining problem in which the comprehensiveness requirement is dropped.

More formally, a *KS-bargaining game* (C, d) is such that $\emptyset \neq C \subset \mathbb{R}^n$, $d \in C$ and

(i) C is compact and convex.
(ii) There is an $x^0 \in C$ with $x^0 > d$.

Let B^N denote the class of all KS-bargaining problems on N. For a (bargaining) solution concept $\gamma: B^N \rightarrow \mathbb{R}^N$, Kalai and Smorodinsky (1975) and Roth (1979) considered the following four properties.

Pareto optimality: For all $(C, d) \in B^N$ there does not exist an $x \in C$ with

$x \geq \gamma(C, d)$ and $x \neq \gamma(C, d)$.

Symmetry: If $d_i = d_j$ for all $i, j \in N$ and $C \subset \mathbb{R}^n$ is such that $(c_i)_{i \in N} \in C$ implies that $(c_{\pi(i)})_{i \in N} \in C$ for each permutation π: $N \rightarrow N$, then

$\gamma_i(C, d) = \gamma_j(C, d)$ for all $i, j \in N$.

(Restricted) Monotonicity: For all (C, d), $(D, d) \in B^N$ with $C \subset D$ and $u(C, d) = u(D, d)$ it holds that

$$\gamma(C, d) \leq \gamma(D, d).$$

Invariance: For all $(C, d) \in B^N$ and each affine function $f: \mathbb{R}^n \to \mathbb{R}^n$ with $f(x) = \alpha x + \beta$, $x \in \mathbb{R}^n$, for some $\alpha \in \mathbb{R}^n_{++}$ and $\beta \in \mathbb{R}^n$, it holds that

$$\gamma(f(C), f(d)) = f(\gamma(C, d)).$$

It was shown that the KS-solution is the unique solution on the class of 2-person KS-bargaining problems which satisfies (the restriction of) these four properties. However, by means of the following example, Roth (1979) showed that there could be no solution which satisfies the first three properties for general n-person KS-bargaining problems.

Example 1: Let $N = \{1, 2, 3\}$, $C = Conv(\{0, 0, 0), (0, 1, 1), (1, 0, 1)\})$ and $d = (0, 0, 0)$. Suppose there is a solution γ that satisfies Pareto-optimality, symmetry and monotonicity. Then, by Pareto-optimality

$$\gamma(C, d) \in Conv(\{0, 1, 1), (1, 0, 1)\}).$$

If a second bargaining problem (D, d) is defined by

$$D := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 \leq 2 \text{ and } 0 \leq x_i \leq 1 \text{ for all } i \in \{1, 2, 3\}\},$$

then symmetry and Pareto optimality imply $\gamma(D, d) = (\frac{2}{3}, \frac{2}{3}, \frac{2}{3})$.

However, since $C \subset D$ and $u(C, d) = u(D, d) = (1, 1, 1)$, monotonicity would imply that $\gamma(C, d) \leqq (\frac{2}{3}, \frac{2}{3}, \frac{2}{3})$, giving a contradiction. Note that the Kalai-Smorodinsky solution of (C, d) equals $(0, 0, 0)$.

Modifying the problem (C, d) to fit in the formalism of bargaining problems introduced in Section 2 by considering $\mathrm{Comp}(C)$ instead of C (cf. Peters and Tijs (1984)), one finds that the KS-solution (i.e. the compromise value of the associated NTU-game) equals $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.

As we will see in Lemma 3 and Theorem 4 below, comprehensiveness together with a weakening of Pareto-optimality (and symmetry) will allow an axiomatic extension of the 2-person KS-solution to the compromise value for a class of NTU-games.

Let $F: \mathscr{C}^N \to \mathbb{R}^N$ be a solution concept for compromise admissible NTU-games. The rule F is called *efficient* if

$$F(V) \in V(N) \setminus \mathrm{Dom}(V(N)) \text{ for all } V \in \mathscr{C}^N. \tag{10}$$

F is *symmetric* if for all $V \in \mathscr{C}^N$ and $i, j \in N$

$$k_i(V) = k_j(V),\ K_i(V) = K_j(V) \Rightarrow F_i(V) = F_j(V). \tag{11}$$

Further F satisfies *monotonicity* if for all $V, W \in \mathscr{C}^N$

$$k(V)=k(W),\ K(V)=K(W),\ V(N) \subset W(N) \Rightarrow F(V) \leq F(W). \tag{12}$$

For an NTU-game (N, V), $\alpha \in \mathbb{R}^N_{++}$ and $\beta \in \mathbb{R}^N$ we can define the NTU-game $(N, \alpha * V+\beta)$ by

$$(\alpha * V+\beta)(S) := \alpha_S * V(S) + \{\beta_S\} \text{ for all } S \in 2^N \setminus \{\emptyset\}.$$

One easily verifies that for compromise admissible V, $\alpha * V+\beta$ is compromise admissible too. The rule F satisfies *invariance* if for all $V \in \mathscr{C}^N$, $\alpha \in \mathbb{R}^N_{++}$ and $\beta \in \mathbb{R}^N$,

$$F(\alpha * V+\beta) = \alpha * F(V) + \beta. \tag{13}$$

Note that both efficiency and symmetry establish weaker conditions than Pareto optimality and symmetry for the special case of bargaining problems, while monotonicity and invariance are immediate extensions of their "bargaining-counterparts".

Lemma 3: The compromise value T: $\mathscr{C}^N \to \mathbb{R}^N$ satisfies efficiency, symmetry, monotonicity and invariance.

Proof: Symmetry and monotonicity are obvious. Invariance follows from the fact that for all $V \in \mathscr{C}^N$, $\alpha \in \mathbb{R}^N_{++}$ and $\beta \in \mathbb{R}^N$ it holds that

$$K(\alpha * V+\beta) = \alpha * K(V) + \beta \text{ and } k(\alpha * V+\beta) = \alpha * k(V) + \beta.$$

Let $V \in \mathscr{C}^N$. We show efficiency by proving that $T(V) \in V(N) \setminus Dom(V(N))$. By definition we have $T(V) \in V(N)$. Suppose $T(V) \in Dom(V(N))$. Then there is an $y \in V(N)$ such that $y > T(V)$. Comprehensiveness and the definition of $T(V)$ imply that $T(V) = K(V)$. Then however $K(V) \in Dom(V(N))$ which contradicts the fact that $V \in \mathscr{C}^N$. □

Let Γ^N denote the class of all compromise admissible NTU-games that satisfy the following three properties:

(i) $V(N)$ is convex and $\{x \in V(N) \mid x \geq k(V)\}$ is *non-level* in the sense that its boundary has no segments parallel to a coordinate hyperplane (cf. Aumann (1985a)).
(ii) $k(V) < K(V)$.
(iii) $(k_{N \setminus \{i\}}(V), K_i(V)) \in V(N)$ for all $i \in N$.

Convexity and non-levelness are important assumptions in the axiomatic characterizations of the Shapley NTU value and the Harsanyi solution (see Aumann (1985a) and Hart (1985a)). Condition (iii) requires that each players' utopia payoff cannot be "too big" relative to the minimal right payoffs of the other players. For example, the class Γ^N contains all NTU-games satisfying condition (i) that correspond to bargaining problems.

Restricting the domain of a solution concept to Γ^N, the four properties discussed above establish an axiomatic characterization of the compromise value.

Theorem 4: The compromise value T is the unique rule on Γ^N that satisfies efficiency, symmetry, monotonicity and invariance.

Proof: (a) Using Lemma 3 it is seen that the compromise value satisfies the four properties. With respect to invariance one should note that $V \in \Gamma^N$, $\alpha \in \mathbb{R}^N_{++}$ and $\beta \in \mathbb{R}^N$ imply $\alpha * V + \beta \in \Gamma^N$.
(b) Let $F: \Gamma^N \to \mathbb{R}^N$ satisfy the four properties and let $W \in \Gamma^N$. We prove that $F(W) = T(W)$. Defining $V := W - k(W)$ it follows that $k(V) = 0$ and, since $k(V) < K(V)$, $\lambda \in \mathbb{R}^N_{++}$ with $\lambda_i := (K_i(V))^{-1}$ for $i \in N$ is well-defined.

Obviously, $k(\lambda * V) = \lambda * k(V) = 0$ and $K(\lambda * V) = \lambda * K(V) = e_N$, so the compromise value $T(\lambda * V)$ lies on the line segment connecting the origin and e_N. Moreover, by (i) and (iii) in the definition of Γ^N we have that $\mathrm{Conv}(\{e_1, \ldots, e_n\}) \subset \lambda * V(N)$, so efficiency of T implies $T(\lambda * V) \geqq \frac{1}{n} e_N$. Non-levelness implies $T(\lambda * V) < e_N$.

Now consider the NTU-game U defined by

$$U(S) := \begin{cases} \{x \in \mathbb{R}^S \mid x \leq 0\} & \text{if } S \neq \emptyset,\ S \subsetneqq N \\ \mathrm{Comp}(Conv(\{e_1, \ldots, e_n, T(\lambda * V)\})) & \text{if } S = N. \end{cases}$$

Obviously, since $T(\lambda * V) \leq e_N$, $K(U) = e_N$ and, consequently, $k(U) = 0$. Hence U is compromise admissible and (ii) and (iii) in the definition of Γ^N are satisfied. Trivially, $U(N)$ is convex and, using the fact that $T(\lambda * V) < e_N$, one readily verifies that $\{x \in U(N), x \geq 0\}$ is non-level which implies that $U \in \Gamma^N$.

Using symmetry of F it follows that $F_i(U) = F_j(U)$ for all $i, j \in N$, so by efficiency and the construction of $U(N)$ we have

$$F(U) = T(\lambda * V).$$

Clearly $U(N) \subset \lambda * V(N)$, and monotonicity implies

$$F(U) \leq F(\lambda * V), \text{ i.e. } T(\lambda * V) \leq F(\lambda * V).$$

But then, using efficiency and non-levelness, $T(\lambda * V) = F(\lambda * V)$.

Hence, by invariance of both T and F, $T(V) = F(V)$ and $T(W) = F(W)$ which finishes the proof. □

We now provide an alternative axiomatic characterization of the compromise value based on the characterization of the TU τ-value of Tijs (1987).

Let F: $\mathscr{C}^N \to \mathbb{R}^N$. The rule F is said to have the *minimal right property* if

$$F(V) = k(V) + F(V - k(V)) \text{ for all } V \in \mathscr{C}^N, \tag{14}$$

and F has the *restricted proportionality property* if $F(V)$ is a multiple of the utopia vector $K(V)$ for all $V \in \mathscr{C}^N$ with $k(V) = 0$.

Efficiency together with these two properties characterize the compromise value on $\tilde{\mathscr{C}}^N$, where $\tilde{\mathscr{C}}^N$ is the class of all compromise admissible NTU-games (N, V) for which $\{x \in V(N) \mid x \geq k(V)\}$ is non-level.

Theorem 5: The compromise value T is the unique rule on $\tilde{\mathscr{C}}^N$ that satisfies efficiency, the minimal right property and the restricted proportionality property.

Proof: (a) Obviously, the compromise value $T: \tilde{\mathscr{C}}^N \to \mathbb{R}^N$ satisfies the minimal right property and restricted proportionality. For efficiency we refer to Lemma 3.
(b) Let $F: \tilde{\mathscr{C}}^N \to \mathbb{R}^N$ satisfy the three properties stated in the theorem. Let $V \in \tilde{\mathscr{C}}^N$. We prove that $F(V) = T(V)$. Using the minimal right property, we deduce that $F(V) = k(V) + F(V - k(V))$. Since $k(V - k(V)) = 0$, the restricted proportionality property implies there is a $\lambda \in \mathbb{R}$ such that $F(V) = k(V) + \lambda K(V - k(V)) = \lambda K(V) + (1 - \lambda) k(V)$.
Using non-levelness of $\{x \in V(N) \mid x \geq k(V)\}$, efficiency of F implies that $\lambda = \lambda_V$ with λ_V as in (9). Hence, $F(V) = T(V)$. □

5 Examples

In this section, we compute the compromise value for two well known examples. The first is found in Roth (1980) and the second is a modification of an example of Shafer (1980) that is found in Hart and Kurz (1983). These examples have provoked an interesting discussion in the literature focusing on the interpretation of the Shapley NTU value (defined in Shapley (1969) and axiomatized in Aumann (1985a)) and the Harsanyi solution (defined in Harsanyi (1963) and axiomatized in Hart (1985a)). We will assume that the reader is familiar with these two concepts and the detailed discussions in Aumann (1985b), Roth (1986), Aumann (1986) and Hart (1985b).

Example 2: Let $N = \{1, 2, 3\}$. For a parameter p with $0 \leq p \leq \frac{1}{2}$, the NTU-game (N, V_p) is defined by (the subscripts denote players):

$$\begin{aligned}
&V_p(\{i\}) = \{a_i \in \mathbb{R} \mid a_i \leq 0\} \quad (i \in N)\\
&V_p(\{1, 2\}) = \{(a_1, a_2) \in \mathbb{R}^2 \mid (a_1, a_2) \leq (\tfrac{1}{2}, \tfrac{1}{2})\}\\
&V_p(\{1, 3\}) = \{(a_1, a_3) \in \mathbb{R}^2 \mid (a_1, a_3) \leq (p, 1-p)\}\\
&V_p(\{2, 3\}) = \{(a_2, a_3) \in \mathbb{R}^2 \mid (a_2, a_3) \leq (p, 1-p)\}\\
&V_p(\{1, 2, 3\}) = \{a = (a_1, a_2, a_3) \in \mathbb{R}^3 \mid a \leq b \text{ for some}\\
&\qquad\qquad b \in Conv\{(\tfrac{1}{2}, \tfrac{1}{2}, 0), (p, 0, 1-p), (0, p, 1-p)\}\}.
\end{aligned}$$

If $0 \leq p < \frac{1}{2}$, then

$$C(V_p) = Conv\{(\tfrac{1}{2}, \tfrac{1}{2}, 0), (\tfrac{1}{2}, p, 0)\} \cup Conv\{(\tfrac{1}{2}, \tfrac{1}{2}, 0), (p, \tfrac{1}{2}, 0)\}$$

and

$$K(V_p)=(\tfrac{1}{2},\tfrac{1}{2},0) \text{ and } k(V_p)=(p,p,0).$$

Further, $(N, V_{\frac{1}{2}})$ is a symmetric game with

$$C(V_{\frac{1}{2}})=\{(\tfrac{1}{2},\tfrac{1}{2},0),(\tfrac{1}{2},0,\tfrac{1}{2}),(0,\tfrac{1}{2},\tfrac{1}{2})\},\ K(V_{\frac{1}{2}})=(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}) \text{ and } k(V_{\frac{1}{2}})=(0,0,0).$$

For the games V_p with $0\le p<\frac{1}{2}$, it follows that the compromise value $T(V_p)$ equals the core element $K(V_p)=(\frac{1}{2},\frac{1}{2},0)$. Further, one finds that $\lambda_{V_{1/2}}=\frac{2}{3}$. So, for $p=\frac{1}{2}$, the compromise value is equal to $(\frac{1}{3},\frac{1}{3},\frac{1}{3})$.

In Table 1, we compare the compromise value, the Shapley NTU value and the Harsanyi NTU value for Example 2.

Table 1

	$p=0$	$0<p<\frac{1}{2}$	$p=\frac{1}{2}$
Shapley	$\{(\frac{1}{2},\frac{1}{2},0),(\frac{1}{3},\frac{1}{3},\frac{1}{3})\}$	$(\frac{1}{3},\frac{1}{3},\frac{1}{3})$	$(\frac{1}{3},\frac{1}{3},\frac{1}{3})$
Harsanyi	$(\frac{1}{2},\frac{1}{2},0)$	$(\frac{1}{2}-\frac{p}{3},\frac{1}{2}-\frac{p}{3},\frac{2p}{3})$	$(\frac{1}{3},\frac{1}{3},\frac{1}{3})$
Compromise	$(\frac{1}{2},\frac{1}{2},0)$	$(\frac{1}{2},\frac{1}{2},0)$	$(\frac{1}{3},\frac{1}{3},\frac{1}{3})$

If $p=\frac{1}{2}$, the game $V_{\frac{1}{2}}$ is completely symmetric and all three solutions yield the natural outcome $(\frac{1}{3},\frac{1}{3},\frac{1}{3})$. For $0\le p<\frac{1}{2}$, Roth argues that $(\frac{1}{2},\frac{1}{2},0)$ is the most "reasonable" outcome since it is the unique (strong) core outcome, it yields to players 1 and 2 the highest possible payoffs that they can attain in the game V_p and players 1 and 2 can attain their payoffs without player 3. Aumann has counterargued that if coalitions can form randomly, then player 3's expected payoff could be positive if players 1 and 2 are willing to "settle" when they are paired with player 3 in order to avoid being shut out later. As p decreases, the negotiating position of player 3 becomes weaker since 1 and 2 have little to lose if they fail to strike a bargain with 3. Thus, player 3's payoff is arguably positive but decreasing in p. This is precisely how the Harsanyi solution behaves. On the other hand, the Shapley NTU value and the compromise value are "extreme" outcomes. Informally, the Shapley NTU value treats player 3 as if he were "as powerful" as players 1 and 2 (i.e. as if $p=\frac{1}{2}$) even when p is close to 0. On the other hand, the compromise value treats player 3 as if he were "powerless" relative to players 1 and 2 (i.e. as if $p=0$) even if p is close to $\frac{1}{2}$. Mathematically, the TU game from which the Shapley NTU value is computed treats the players symmetrically if $0<p\le\frac{1}{2}$. At the other extreme, the utopia and the minimal right payoffs for player 3 in V_p are 0 if $0\le p<\frac{1}{2}$.

Example 3: Consider an exchange market with three traders and two commodities, where the initial endowment $\omega_i\in\mathbb{R}^2_+$ and the utility function $u_i\colon\mathbb{R}^2_+\to\mathbb{R}$ of trader $i\in\{1,2,3\}$ are given by

$$\omega_1=(1-\varepsilon,0),\ \omega_2=(0,1-\varepsilon),\ \omega_3=(\varepsilon,\varepsilon)$$
$$u_1(c_1,c_2)=u_2(c_1,c_2)=\min\{c_1,c_2\} \text{ and } u_3(c_1,c_2)=\tfrac{1}{2}(c_1+c_2) \text{ for all } (c_1,c_2)\in\mathbb{R}^2_+$$

for some $0 \leq \varepsilon < \frac{1}{5}$.

This exchange market corresponds to an NTU-game (N, V) with $N = \{1, 2, 3\}$ and

$$V_\varepsilon(S) := \{a \in \mathbb{R}^S \mid \exists_{f: S \to \mathbb{R}^2} \forall_{i \in S} : u_i(f(i)) \geq a_i, \sum_{j \in S} f(j) = \sum_{j \in S} \omega_j\} \text{ for all } S \in 2^N \setminus \{\emptyset\}.$$

So, in particular, with subscripts representing players,

$$V_\varepsilon(\{1\}) = \{a_1 \in \mathbb{R} \mid a_1 \leq 0\},\ V(\{2\}) = \{a_2 \in \mathbb{R} \mid a_2 \leq 0\},\ V(\{3\}) = \{a_3 \in \mathbb{R} \mid a_3 \leq \varepsilon\},$$
$$V_\varepsilon(\{1, 2\}) = \{(a_1, a_2) \in \mathbb{R}^2 \mid a_1 + a_2 \leq 1 - \varepsilon,\ a_1 \leq 1 - \varepsilon,\ a_2 \leq 1 - \varepsilon\},$$
$$V_\varepsilon(\{1, 3\}) = \{(a_1, a_3) \in \mathbb{R}^2 \mid a_1 + a_3 \leq \tfrac{1}{2} + \tfrac{1}{2}\varepsilon,\ a_1 \leq \varepsilon,\ a_3 \leq \tfrac{1}{2} + \tfrac{1}{2}\varepsilon\},$$
$$V_\varepsilon(\{2, 3\}) = \{(a_2, a_3) \in \mathbb{R}^2 \mid a_2 + a_3 \leq \tfrac{1}{2} + \tfrac{1}{2}\varepsilon,\ a_2 \leq \varepsilon,\ a_3 \leq \tfrac{1}{2} + \tfrac{1}{2}\varepsilon\},$$
$$V_\varepsilon(\{1, 2, 3\}) = \{(a_1, a_2, a_3) \in \mathbb{R}^3 \mid a_1 + a_2 + a_3 \leq 1,\ a_1 \leq 1,\ a_2 \leq 1,\ a_3 \leq 1\}.$$

One can check that

$$C(V_\varepsilon) = \{(a_1, a_2, a_3) \in \mathbb{R}^3 \mid a_1 + a_2 + a_3 = 1,\ a_1 \geq \varepsilon,\ a_2 \geq \varepsilon,\ a_3 = \varepsilon\}$$

and that $K(V_\varepsilon) = (1 - 2\varepsilon, 1 - 2\varepsilon, \varepsilon)$ and $k(V_\varepsilon) = (\varepsilon, \varepsilon, \varepsilon)$. Therefore, $T(V_\varepsilon) = (\frac{1}{2} - \frac{\varepsilon}{2}, \frac{1}{2} - \frac{\varepsilon}{2}, \varepsilon)$.

We compare the Shapley NTU value, the Harsanyi NTU value and the Compromise value for example 3 in Table 2 below.

Table 2

Shapley	$(\frac{5}{12} - \frac{5\varepsilon}{12}, \frac{5}{12} - \frac{5\varepsilon}{12}, \frac{1}{6} + \frac{5\varepsilon}{6})$
Harsanyi	$(\frac{1}{2} - \frac{5\varepsilon}{6}, \frac{1}{2} - \frac{5\varepsilon}{6}, \frac{5\varepsilon}{3})$
Compromise	$(\frac{1}{2} - \frac{\varepsilon}{2}, \frac{1}{2} - \frac{\varepsilon}{2}, \varepsilon)$

Shafer has argued that in this example of a pure exchange economy, the Shapley NTU value is not reasonable because player 3 receives a utility level of at least $\frac{1}{6}$ even if $\varepsilon = 0$ because of the utility producing properties of player 3's utility function. However, the same "safety vs. coordination" argument as given above can be applied to this example to justify a payoff to player 3 that is positive but decreasing in ε. Both the Harsanyi payoff and the compromise value payoff are consistent with such an argument with the compromise value giving 1 and 2 more (and 3 less) than they receive in the Harsanyi value.

6 The NTU τ-Value

In Shapley (1969) the λ-transfer TU-game associated with an NTU-game is introduced and the NTU (Shapley)-value is obtained from the Shapley value of these games. Analogously, this section introduces the NTU τ-value by means of the τ-value of quasi-balanced λ-transfer games.

Let (N, V) be an NTU-game. Define $\Delta_N := \{\lambda \in \mathbb{R}^N \mid \lambda \geq 0, \sum_{i \in N} \lambda_i = 1\}$. A vector $\lambda \in \Delta_N$ is called *V-feasible* if $\sup\{\sum_{i \in S} \lambda_i a_i \mid a \in V(S)\} < \infty$ for all $S \in 2^N \setminus \{\emptyset\}$. For each V-feasible vector $\lambda \in \Delta_N$ the TU-game (N, v_λ) with

$$v_\lambda(\emptyset) := 0, \; v_\lambda(S) := \sup\{\sum_{i \in S} \lambda_i a_i \mid a \in V(S)\} \text{ for } S \in 2^N \setminus \{\emptyset\} \tag{15}$$

is called a *λ-transfer game* corresponding to V.

If for all V-feasible λ the corresponding λ-transfer games are quasi-balanced, then the game V is called *τ-admissible*. By $\mathscr{A}^N$ we denote the class of all τ-admissible NTU-games with player set N. For $V \in \mathscr{A}^N$ the *NTU τ-value* $\tau(V) \subset \mathbb{R}^N$ is defined by

$$\tau(V) := \{x \in V(N) \mid \text{there is a } V\text{-feasible } \lambda \in \Delta_N \text{ such that } \tau(v_\lambda) = \lambda * x\}. \tag{16}$$

For TU-games the NTU τ-value coincides with the τ-value. Consider an NTU-game (N, V) that arises from a quasi-balanced TU-game (N, v). Obviously, $\lambda = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$ is the unique V-feasible vector in Δ_N. Further, since $v_\lambda(S) = \frac{1}{n} v(S)$ for all $S \in 2^N$, $\tau(v_\lambda) = \frac{1}{n}\tau(v) = \lambda * \tau(v)$. Hence, $\tau(V) = \{\tau(v)\}$.

For two-person bargaining problems, the NTU τ-value and the Nash bargaining solution (cf. Nash (1950)) coincide. Let $(\{1, 2\}, V)$ correspond to a bargaining problem (C, d). Obviously, V is τ-admissible. Since for each (quasi-balanced) two-person TU-game, the Shapley value and the τ-value coincide, it follows that the NTU-value and the NTU τ-value are the same for each (two-person) bargaining game. Moreover, since the NTU-value coincides with the Nash bargaining solution (cf. Shapley (1969)), this also holds for the NTU τ-value.

To further illustrate the NTU τ-value, reconsider the game V_ε of Example 3 in Section 5. One easily checks that each $\lambda \in \Delta_N$ is V_ε-feasible. The corresponding λ-transfer games are denoted by $v_{\varepsilon,\lambda}$.

Let $\lambda \in \Delta_N$ and $x \in V_\varepsilon(N)$ be such that $\lambda * x = \tau(v_{\varepsilon,\lambda})$.
Suppose there exists a player $i \in N$ such that $\lambda_i < \max_{j \in N} \lambda_j$. Since $v_{\varepsilon,\lambda}(N) = \max_{j \in N} \lambda_j$ and $\sum_{j \in N} \tau_j(v_{\varepsilon,\lambda}) = v_{\varepsilon,\lambda}(N)$, one finds that $x_i = 0$ and, consequently $\tau_i(v_{\varepsilon,\lambda}) = 0$. Distinguishing cases, some calculation shows that $M_i(v_{\varepsilon,\lambda}) > 0$, $m_i(v_{\varepsilon,\lambda}) \geq 0$ and $\sum_{j \in N} m_j(v_{\varepsilon,\lambda}) \neq v_{\varepsilon,\lambda}(N)$. However, since this should imply that $\tau_i(v_{\varepsilon,\lambda}) > 0$, we arrive at a contradiction. We may conclude that $\lambda = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Then $v_{\varepsilon,\lambda}$ is given by

$$v_{\varepsilon,\lambda}(\emptyset) = 0, \; v_{\varepsilon,\lambda}(\{1\}) = v_{\varepsilon,\lambda}(\{2\}) = 0, \; v_{\varepsilon,\lambda}(\{3\}) = \tfrac{1}{3}\varepsilon, \; v_{\varepsilon,\lambda}(\{1, 2\}) = \tfrac{1}{3} - \tfrac{1}{3}\varepsilon,$$
$$v_{\varepsilon,\lambda}(\{1, 3\}) = v_{\varepsilon,\lambda}(\{2, 3\}) = \tfrac{1}{6} + \tfrac{1}{6}\varepsilon \text{ and } v_{\varepsilon,\lambda}(\{1, 2, 3\}) = \tfrac{1}{3}.$$

Hence, $\tau(v_{\varepsilon,\lambda}) = M(v_{\varepsilon,\lambda}) = m(v_{\varepsilon,\lambda}) = (\frac{1}{6} - \frac{1}{6}\varepsilon, \frac{1}{6} - \frac{1}{6}\varepsilon, \frac{1}{3}\varepsilon)$ and $\tau(V_\varepsilon) = \{(\frac{1}{2} - \frac{1}{2}\varepsilon, \frac{1}{2} - \frac{1}{2}\varepsilon, \varepsilon)\}$.
Note that the compromise value and the (unique) NTU τ-value of V_ε coincide.

An NTU-game (N, V) is called *zero-adjusted* if $v(i) \geq 0$ for all $i \in N$ and *convexly and compactly generated* if, for each $S \in 2^N \setminus \{\emptyset\}$ there exists a convex and compact set $C(S) \subset \mathbb{R}^S$ such that

$$V(S) = \{a \in \mathbb{R}^S \mid \exists_{c \in C(S)}: a \leq c\}. \tag{17}$$

Note that the Shafer game of example 3 satisfies these two properties.

Using the same line of argument as in the existence proof of the NTU-value given by Shapley (1969), one can show

Theorem 6: Let the NTU-game (N, V) be τ-admissible, zero-adjusted and convexly and compactly generated. Then $\tau(V) \neq \emptyset$.

Let (N, V) be an NTU-game. Note that the NTU τ-value is defined only if for *all* V-feasible $\lambda \in \Delta_N$ the corresponding λ-transfer games are quasi-balanced. However, the definition readily can be extended to a larger class of games by requiring that only *some* feasible $\lambda \in \Delta_N$ give rise to quasi-balanced λ-transfer games. More specifically, we introduce

$$\begin{aligned}\tau^*(V) := \{x \in V(N) \mid \text{ there is a } V\text{-feasible } \lambda \in \Delta_N \text{ such that } v_\lambda \text{ is} \\ \text{quasi-balanced and } \lambda * x = \tau(v_\lambda)\}.\end{aligned} \tag{18}$$

Obviously, if V is τ-admissible, then $\tau^*(V) = \tau(V)$. Using this extended definition, the NTU τ-value can be calculated for the Roth games V_p of example 2 for $0 \leq p \leq \frac{1}{2}$. Since V_p is compactly generated, each $\lambda \in \Delta_N$ is V_p-feasible. The corresponding λ-transfer games $v_{p,\lambda}$ are given by $v_{p,\lambda}(\{i\}) = 0$ for all $i \in N$,

$$\begin{aligned}&v_{p,\lambda}(\{1, 2\}) = \tfrac{1}{2}(\lambda_1 + \lambda_2),\ v_{p,\lambda}(\{1, 3\}) = p\lambda_1 + (1-p)\lambda_3, \\ &v_{p,\lambda}(\{2, 3\}) = p\lambda_2 + (1-p)\lambda_3 \text{ and} \\ &v_{p,\lambda}(N) = \max\{\tfrac{1}{2}(\lambda_1 + \lambda_2),\ p\lambda_1 + (1-p)\lambda_3,\ p\lambda_2 + (1-p)\lambda_3\}.\end{aligned}$$

Note that V_p is not τ-admissible because for $\bar{\lambda} = (\frac{1}{10}, \frac{1}{10}, \frac{8}{10})$ we have that

$$M_1(v_{p,\bar{\lambda}}) = 0 < \tfrac{1}{10} = m_1(v_{p,\bar{\lambda}}),$$

which implies that $v_{p,\bar{\lambda}}$ is not quasi-balanced. With respect to τ^* it can be shown that

$$\tau^*(V_p) = \{(\tfrac{1}{2}, \tfrac{1}{2}, x_3) \mid x_3 \leq 0\} \cup \{x \in \mathbb{R}^N \mid \sum_{j \in N} x_j \leq 1,\ x_1 \leq p,\ x_2 \leq p,\ x_3 = 1-p\}$$

for all $0 \leqq p < \frac{1}{2}$, and

$$\begin{aligned}\tau^*(V_{1/2}) = \{x \in \mathbb{R}^N \mid x_1 = \tfrac{1}{2},\ x_2 + x_3 \leq \tfrac{1}{2}\} \cup \{x \in \mathbb{R}^N \mid x_2 = \tfrac{1}{2},\ x_1 + x_3 \leq \tfrac{1}{2}\} \\ \cup \{x \in \mathbb{R}^N \mid x_3 = \tfrac{1}{2},\ x_1 + x_2 \leq \tfrac{1}{2}\}.\end{aligned}$$

Remark: Should one restrict attention to *positive* V-feasible vectors λ only, there does not exist an NTU τ-value in case $p=\frac{1}{2}$ and, for $0 \leq p < \frac{1}{2}$, there is a unique NTU τ-value $(\frac{1}{2}, \frac{1}{2}, 0)$.

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